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The torque required for a steady rotation of a disk in a quiescent fluid

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Abstract. For large values of the Reynolds number Re two terms of the asymptotic series for the torque have been calculated. They are of order $\mathrm{Re}^{-1/2}$ and $\mathrm{Re}^{-13/14}$, respectively. The second term has been obtained after investigation of the double-deck structure which is present near the edge of the disk over a length of order $\mathrm{Re}^{-3/7}$.

1. Introduction

In a previous publication [1] the authors have considered the flow outside a rotating disk of finite radius a on the basis of the boundary layer equations. Near the edge of the disk this led to a singularity in the axial velocity w of strength $\mathrm{O}(r-a)^{-2/3}$ for $r \downarrow a$, where r is the radial coordinate. This singularity is due to the sudden change in boundary conditions in the plane of the disk. While for $r < a$ the radial and tangential velocities in this plane are prescribed, for $r > a$ the derivatives in axial direction of these velocities vanish due to symmetry with regard to the disk plane. This sudden change of the boundary conditions in radial direction means that the boundary layer equations lose their validity locally. The situation is analogous to that investigated by Smith and Duck [2] and by Smith [3] for a jet streaming along a suddenly ending wall. They found that due to the sudden change in boundary conditions a double-deck structure arises in the boundary layer over a length of $\mathrm{O}(\mathrm{Re}^{-3/7})$ at both sides of the edge of the wall. The same occurs at the edge of the rotating disk, though there is an additional tangential velocity. The fact that there is no triple deck is due to the absence of a driving external mainstream, which makes that no pressure variations outside the original boundary layer occur and therefore no upper deck exists. In the lower deck the tangential shear stress is numerically modified, leading to an additional contribution to the torque required for a steady rotation of the disk. This contribution is $\mathrm{O}(\mathrm{Re}^{-13/14})$. In this way we have obtained two terms of the asymptotic series for the torque in the limit $\mathrm{Re}\to\infty$.

2. The middle deck

For an axially-symmetric system the Navier–Stokes equations are in dimensionless form, see e.g. [4],

$$u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}=-\frac{\partial p}{\partial r}+\mathrm{Re}^{-1}\left\{\frac{\partial^2 u}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{u}{r}\right)+\frac{\partial^2 u}{\partial z^2}\right\},$$

$$u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}=\mathrm{Re}^{-1}\left\{\frac{\partial^2 v}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{v}{r}\right)+\frac{\partial^2 v}{\partial z^2}\right\},$$

$$u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+\mathrm{Re}^{-1}\left\{\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right\},$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ur)+\frac{\partial w}{\partial z}=0,$$

$$u=\frac{1}{r}\frac{\partial\psi}{\partial z},\qquad w=-\frac{1}{r}\frac{\partial\psi}{\partial r},$$

where u, v, w are the radial, tangential and axial components of the velocity, while ψ is the stream function and p the pressure. Lengths have been made dimensionless with the radius a, velocities with Ωa and the pressure with $\rho\Omega^2a^2$. Ω denotes the angular velocity of the disk and ρ the fluid density. The Reynolds number Re is $\Omega a^2/\nu$ with ν the kinematic viscosity coefficient.

After transformation to boundary layer variables,

$$z=\mathrm{Re}^{-1/2}\tilde{z},\qquad w=\mathrm{Re}^{-1/2}\tilde{w},\qquad \psi=\mathrm{Re}^{-1/2}\tilde{\psi},$$

the equations become

$$u\frac{\partial u}{\partial r}+\tilde{w}\frac{\partial u}{\partial\tilde{z}}-\frac{v^2}{r}=-\frac{\partial p}{\partial r}+\mathrm{Re}^{-1}\left\{\frac{\partial^2 u}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{u}{r}\right)\right\}+\frac{\partial^2 u}{\partial\tilde{z}^2},$$

$$u\frac{\partial v}{\partial r}+\tilde{w}\frac{\partial v}{\partial\tilde{z}}+\frac{uv}{r}=\mathrm{Re}^{-1}\left\{\frac{\partial^2 v}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right\}+\frac{\partial^2 v}{\partial\tilde{z}^2},$$

$$u\frac{\partial\tilde{w}}{\partial r}+\tilde{w}\frac{\partial\tilde{w}}{\partial\tilde{z}}=-\mathrm{Re}\frac{\partial p}{\partial\tilde{z}}+\mathrm{Re}^{-1}\left\{\frac{\partial^2\tilde{w}}{\partial r^2}+\frac{1}{r}\frac{\partial\tilde{w}}{\partial r}\right\}+\frac{\partial^2\tilde{w}}{\partial\tilde{z}^2},\tag{2.1}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ur)+\frac{\partial\tilde{w}}{\partial\tilde{z}}=0,$$

$$u=\frac{1}{r}\frac{\partial\tilde{\psi}}{\partial\tilde{z}},\qquad \tilde{w}=-\frac{1}{r}\frac{\partial\tilde{\psi}}{\partial r}.$$

In boundary layer theory all variables u, v, $\tilde{w}$, $\tilde{\psi}$, r and $\tilde{z}$ are $O(\mathrm{Re}^0)$. In the first two equations (2.1) the terms with Re^{-1} are neglected and the third equation simplifies to $\partial p/\partial\tilde{z}=0$. Since it follows, see [1], that $\tilde{w}\to\infty$ for $r\downarrow 1$, this theory is not valid in a small region near $r=1$. Let us suppose that the size in r-direction of this small region is $\mathrm{Re}^{-\alpha}$ with $\alpha>0$. We introduce a new variable r^* in this region by

$$r=1+r^*\,\mathrm{Re}^{-\alpha}\quad\text{or}\quad r^*=\mathrm{Re}^{\alpha}(r-1).\tag{2.2}$$

Then $\partial/\partial r=\mathrm{Re}^{\alpha}(\partial/\partial r^*)$ and it is assumed that in the r^*-region derivatives with respect to r^* are $O(1)$.

We use the following asymptotic expansions for $\mathrm{Re}\to\infty$, $r\to 1$ and r^* fixed:

$$\tilde{\psi}(r,\tilde{z};\mathrm{Re})=\tilde{\psi}_0(r,\tilde{z})+\mu_1(\mathrm{Re})\tilde{\psi}_1(r^*,\tilde{z})+\cdots,$$

$$u(r,\tilde{z};\mathrm{Re})=u_0(r,\tilde{z})+\mu_1(\mathrm{Re})u_1(r^*,\tilde{z})+\cdots,$$

$$v(r,\tilde{z};\mathrm{Re})=v_0(r,\tilde{z})+\nu_1(\mathrm{Re})v_1(r^*,\tilde{z})+\cdots,$$

$$\tilde{w}(r,\tilde{z};\mathrm{Re}) = \tilde{w}_0(r,\tilde{z}) + \mathrm{Re}^{\alpha}\mu_1(\mathrm{Re})\tilde{w}_1(r^*,\tilde{z}) + \cdots,$$
$$p(r,\tilde{z};\mathrm{Re}) = p_0(r,\tilde{z}) + \pi_1(\mathrm{Re})p_1(r^*,\tilde{z}) + \cdots.$$

Here $\tilde{\psi}_0$, u_0, etc. are the solutions of the Navier–Stokes equation for $r<1$. In the notation of [1, 4] these are

$$\tilde{\psi}_0 = \tfrac{1}{2}r^2H(\tilde{z})\,, \qquad v_0 = rG(\tilde{z})\,, \qquad p_0 = \mathrm{Re}^{-1}P(\tilde{z})\,.$$

After substitution of these quantities in (2.1) a set of ordinary differential equations is obtained for G, H and P, which can be solved numerically. Henceforth, these functions will be considered as known, see [4], where the sign of H is taken opposite.

In order to compensate for the singularity of $\tilde{w}$ from boundary layer theory if $r\downarrow 1$, it is necessary that the second term in the expansion of $\tilde{w}$ is more important for $\mathrm{Re}\to\infty$ than the first one. This means that the order of μ_1 will be larger than $\mathrm{Re}^{-\alpha}$ for $\mathrm{Re}\to\infty$. Using (2.2) this allows us to replace r by 1 in the expansions. Moreover, it follows from the third equation (2.1) that also the second term in the expansion of p will be more important than the first one. The expansions then become

$$\begin{aligned}
\tilde{\psi} &= \tfrac{1}{2}H(\tilde{z}) + \mu_1(\mathrm{Re})\tilde{\psi}_1(r^*,\tilde{z}) + \cdots,\\
u &= \tfrac{1}{2}H'(\tilde{z}) + \mu_1(\mathrm{Re})u_1(r^*,\tilde{z}) + \cdots,\\
v &= G(\tilde{z}) + \nu_1(\mathrm{Re})v_1(r^*,\tilde{z}) + \cdots,\\
\tilde{w} &= \mathrm{Re}^{\alpha}\mu_1(\mathrm{Re})\tilde{w}_1(r^*,\tilde{z}) + \cdots,\\
p &= \pi_1(\mathrm{Re})p_1(r^*,\tilde{z}) + \cdots,\\
&\frac{\partial u_1}{\partial r^*} + \frac{\partial \tilde{w}_1}{\partial \tilde{z}} = 0\,, \qquad u_1 = \frac{\partial\tilde{\psi}_1}{\partial\tilde{z}}\,, \qquad \tilde{w}_1 = -\frac{\partial\tilde{\psi}_1}{\partial r^*}\,.
\end{aligned} \tag{2.3}$$

Substitution of the expansions (2.3) into (2.1) yields as most important terms

$$\mathrm{Re}^{\alpha}\mu_1\Big(\frac{1}{2}H'\frac{\partial u_1}{\partial r^*} + \frac{1}{2}H''\tilde{w}_1\Big) = -\mathrm{Re}^{\alpha}\pi_1\frac{\partial p_1}{\partial r^*} + \mathrm{Re}^{2\alpha-1}\mu_1\frac{\partial^2 u_1}{\partial r^{*2}}\,,$$
$$\mathrm{Re}^{\alpha}\nu_1\Big(\frac{1}{2}H'\frac{\partial v_1}{\partial r^*}\Big) + \mathrm{Re}^{\alpha}\mu_1 G'\tilde{w}_1 = \mathrm{Re}^{2\alpha-1}\nu_1\frac{\partial^2 v_1}{\partial r^{*2}}\,,$$
$$\mathrm{Re}^{2\alpha}\mu_1\Big(\frac{1}{2}H'\frac{\partial \tilde{w}_1}{\partial r^*}\Big) = -\mathrm{Re}\,\pi_1\frac{\partial p_1}{\partial\tilde{z}} + \mathrm{Re}^{3\alpha-1}\mu_1\frac{\partial^2\tilde{w}_1}{\partial r^{*2}}\,.$$

Assuming $\alpha<1$ we retain

$$\mu_1\Big(\frac{1}{2}H'\frac{\partial u_1}{\partial r^*} + \frac{1}{2}H''\tilde{w}_1\Big) = -\pi_1\frac{\partial p_1}{\partial r^*}\,,$$
$$\nu_1\frac{1}{2}H'\frac{\partial v_1}{\partial r^*} + \mu_1 G'\tilde{w}_1 = 0\,,$$
$$\mathrm{Re}^{2\alpha-1}\mu_1\frac{1}{2}H'\frac{\partial\tilde{w}_1}{\partial r^*} = -\pi_1\frac{\partial p_1}{\partial\tilde{z}}\,.$$

Next, assuming $\alpha < \frac{1}{2}$ (the smaller α, the more important $\mathrm{Re}^{-\alpha}$, so that we have to search for a minimum value of α), it follows from the last equation that

$$\pi_1 = \mathrm{O}(\mathrm{Re}^{2\alpha-1}\mu_1)\,, \tag{2.4}$$

and hence that the order of π_1 is smaller than the order of μ_1. Moreover, the second equation shows that μ_1 and ν_1 are of equal order. Then the equations become

$$H'\frac{\partial u_1}{\partial r^*} + H''\tilde{w}_1 = 0\,, \qquad \frac{1}{2}H'\frac{\partial v_1}{\partial r^*} + G'\tilde{w}_1 = 0\,, \qquad \frac{1}{2}H'\frac{\partial \tilde{w}_1}{\partial r^*} = -\frac{\partial p_1}{\partial \tilde{z}}\,. \tag{2.5}$$

Combination of the continuity equation in (2.3) with the first equation (2.5) permits integration. This leads to the expressions

$$\begin{aligned} \tilde{w}_1(r^*, \tilde{z}) &= -E'(r^*)H'(\tilde{z})\,, \\ \tilde{\psi}_1(r^*, \tilde{z}) &= E(r^*)H'(\tilde{z}) + K_1(\tilde{z})\,, \\ u_1(r^*, \tilde{z}) &= E(r^*)H''(\tilde{z}) + K_1'(\tilde{z})\,, \\ v_1(r^*, \tilde{z}) &= 2E(r^*)G'(\tilde{z}) + K_2(\tilde{z})\,, \\ p_1(r^*, \tilde{z}) &= -\frac{1}{2}E''(r^*)\int_{\tilde{z}}^{\infty}\{H'(\tilde{z})\}^2\,\mathrm{d}\tilde{z} + E_1(r^*)\,. \end{aligned} \tag{2.6}$$

$E(r^*)$, $E_1(r^*)$, $K_1(\tilde{z})$ and $K_2(\tilde{z})$ are unknown functions introduced by the integrations.

For $r^* \to \infty$ the solutions (2.6) must be matched to the outer Goldstein solution. This is the boundary layer solution [5] valid for $r > 1$ and $\tilde{z} \gg (r-1)^{1/3}$, which has been calculated in [1] for the case of the rotating disk. The solution is given by equations (4.5) of [1]. Substituting

$$\xi = (r-1)^{1/3} = \mathrm{Re}^{-\alpha/3} r^{*1/3}\,,$$

this solution becomes (as $r^* \to \infty$)

$$\begin{aligned} \tilde{\psi}(r^*, \tilde{z}) &= \tfrac{1}{2}H(\tilde{z}) + \tfrac{1}{2}A\,\mathrm{Re}^{-\alpha/3} r^{*1/3} H'(\tilde{z}) + \mathrm{O}(\mathrm{Re}^{-2\alpha/3})\,, \\ u(r^*, \tilde{z}) &= \tfrac{1}{2}H'(\tilde{z}) + \tfrac{1}{2}A\,\mathrm{Re}^{-\alpha/3} r^{*1/3} H''(\tilde{z}) + \mathrm{O}(\mathrm{Re}^{-2\alpha/3})\,, \\ v(r^*, \tilde{z}) &= G(\tilde{z}) + A\,\mathrm{Re}^{-\alpha/3} r^{*1/3} G'(\tilde{z}) + \mathrm{O}(\mathrm{Re}^{-2\alpha/3})\,, \\ \tilde{w}(r^*, \tilde{z}) &= -\tfrac{1}{6}A\,\mathrm{Re}^{2\alpha/3} r^{*-2/3} H'(\tilde{z}) + \mathrm{O}(\mathrm{Re}^{\alpha/3})\,, \end{aligned} \tag{2.7}$$

where A is a constant given in [1].

Comparing (2.7) with (2.3) and using (2.4), we conclude

$$\mu_1 = \mathrm{Re}^{-\alpha/3}\,, \qquad \nu_1 = \mathrm{Re}^{-\alpha/3}\,, \qquad \pi_1 = \mathrm{Re}^{5\alpha/3-1}\,. \tag{2.8}$$

Comparison of (2.7) with (2.3) and (2.6) leads to

$$E(r^*) \sim \tfrac{1}{2}Ar^{*1/3} \quad \text{as } r^* \to \infty\,, \tag{2.9}$$

while $K_1(\tilde{z})$ and $K_2(\tilde{z})$ are both identical to zero. Furthermore

$$E(r^*) \to 0 \quad \text{as } r^* \to -\infty\,, \tag{2.10}$$

since all expressions (2.6) then must vanish.

Summarizing, the expressions for the middle deck become

$$\begin{aligned}
\tilde{\psi}(r, \tilde{z}; \mathrm{Re}) &= \tfrac{1}{2}H(\tilde{z}) + \mathrm{Re}^{-\alpha/3}E(r^*)H'(\tilde{z}) + O(\mathrm{Re}^{-2\alpha/3})\,,\\
u(r, \tilde{z}; \mathrm{Re}) &= \tfrac{1}{2}H'(\tilde{z}) + \mathrm{Re}^{-\alpha/3}E(r^*)H''(\tilde{z}) + O(\mathrm{Re}^{-2\alpha/3})\,,\\
v(r, \tilde{z}; \mathrm{Re}) &= G(\tilde{z}) + 2\,\mathrm{Re}^{-\alpha/3}E(r^*)G'(\tilde{z}) + O(\mathrm{Re}^{-2\alpha/3})\,,\\
\tilde{w}(r, \tilde{z}; \mathrm{Re}) &= -\mathrm{Re}^{2\alpha/3}E'(r^*)H'(\tilde{z}) + O(\mathrm{Re}^{\alpha/3})\,,\\
p(r, \tilde{z}; \mathrm{Re}) &= \mathrm{Re}^{5\alpha/3-1}\left[-\frac{1}{2}\,E''(r^*)\int_{\tilde{z}}^{\infty}\{H'(\tilde{z})\}^2\,\mathrm{d}\tilde{z} + E_1(r^*)\right].
\end{aligned} \tag{2.11}$$

Since for $\tilde{z} = 0$, $r^* > 0$ the solutions (2.11) do not satisfy the boundary conditions $\partial u/\partial\tilde{z} = 0$, $\partial v/\partial\tilde{z} = 0$ and for $\tilde{z} = 0$, $r^* < 0$ neither satisfy $u = 0$, $v = 1$, there must be a lower deck where the viscous terms will be of importance. Equations (2.5) show that the flow in the middle deck is inviscid.

3. The lower deck

In the lower deck the coordinates are

$$r^* = \mathrm{Re}^{\alpha}(r - 1)\,, \qquad z^* = \mathrm{Re}^{\beta}\tilde{z}\,. \tag{3.1}$$

The asymptotic expansions for $\mathrm{Re} \to \infty$, $r \to 1$, $\tilde{z} \to 0$ with r^* and z^* fixed are

$$\begin{aligned}
\tilde{\psi}(r, \tilde{z}; \mathrm{Re}) &= \mathrm{Re}^{-\beta}\mu_1^*(\mathrm{Re})\psi_1^*(r^*, z^*) + \cdots\,,\\
u(r, \tilde{z}; \mathrm{Re}) &= \mu_1^*(\mathrm{Re})u_1^*(r^*, z^*) + \cdots\,,\\
v(r, \tilde{z}; \mathrm{Re}) &= 1 + \nu_1^*(\mathrm{Re})v_1^*(r^*, z^*) + \cdots\,,\\
\tilde{w}(r, \tilde{z}; \mathrm{Re}) &= \mathrm{Re}^{\alpha-\beta}\mu_1^*(\mathrm{Re})w_1^*(r^*, z^*) + \cdots\,,\\
p(r, \tilde{z}; \mathrm{Re}) &= \pi_1^*(\mathrm{Re})p_1^*(r^*, z^*) + \cdots\,.
\end{aligned} \tag{3.2}$$

It follows from (2.1), (3.1) and (3.2) that

$$u = \frac{1}{r}\,\frac{\partial\tilde{\psi}}{\partial\tilde{z}} = \mu_1^*(\mathrm{Re})\,\frac{\partial\psi_1^*}{\partial z^*}\,, \quad \text{hence} \quad u_1^* = \frac{\partial\psi_1^*}{\partial z^*}\,,$$

$$\tilde{w} = -\frac{1}{r}\,\frac{\partial\tilde{\psi}}{\partial r} = -\mathrm{Re}^{\alpha-\beta}\mu_1^*(\mathrm{Re})\,\frac{\partial\psi_1^*}{\partial r^*}\,, \quad \text{hence} \quad w_1^* = -\frac{\partial\psi_1^*}{\partial r^*}\,.$$

The continuity equation is

$$\frac{\partial u_1^*}{\partial r^*} + \frac{\partial w_1^*}{\partial z^*} = 0\,.$$

For $r^* \to \infty$ the solution (3.2) must be matched to the inner Goldstein solution, which is the boundary layer solution valid for $r > 1$, while $\tilde{z}/(r-1)^{1/3}$ is not too large. This solution is given by equations (3.3) of [1] as

$$u = \mathrm{Re}^{-\alpha/3} r^{*1/3} f_0'\left(\frac{\mathrm{Re}^{-\beta} z^*}{\mathrm{Re}^{-\alpha/3} r^{*1/3}}\right) + \cdots ,$$
$$v = 1 + \mathrm{Re}^{-\alpha/3} r^{*1/3} g_0\left(\frac{\mathrm{Re}^{-\beta} z^*}{\mathrm{Re}^{-\alpha/3} r^{*1/3}}\right) + \cdots , \tag{3.3}$$

Comparing with (3.2), we see that

$$\mu_1^* = \mathrm{Re}^{-\alpha/3} , \qquad \beta = \tfrac{1}{3}\alpha , \qquad \nu_1^* = \mathrm{Re}^{-\alpha/3} ,$$
$$r^* \to \infty \begin{cases} u_1^*(r^*, z^*) \sim r^{*1/3} f_0'\left(\dfrac{z^*}{r^{*1/3}}\right) , \\ v_1^*(r^*, z^*) \sim r^{*1/3} g_0\left(\dfrac{z^*}{r^{*1/3}}\right) . \end{cases} \tag{3.4}$$

Since the pressure in the middle deck is $O(\mathrm{Re}^{5\alpha/3-1})$ and this must be matched to the pressure in the lower deck, we also have

$$\pi_1^* = \mathrm{Re}^{5\alpha/3-1} . \tag{3.5}$$

Considering now in the lower deck the orders of magnitude of the various terms of the first equation of motion (2.1) we find

$$u\,\frac{\partial u}{\partial r} , \ \tilde{w}\,\frac{\partial u}{\partial \tilde{z}} , \ \frac{\partial^2 u}{\partial \tilde{z}^2} = O(\mathrm{Re}^{\alpha/3}) , \qquad \frac{v^2}{r} = O(\mathrm{Re}^0) ,$$
$$\frac{\partial p}{\partial r} = O(\mathrm{Re}^{8\alpha/3-1}) , \qquad \mathrm{Re}^{-1}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\,\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) = O(\mathrm{Re}^{5\alpha/3-1}) .$$

Since the pressure differences in the double deck are generated by velocity differences in the lower deck, these must be of the same order which means

$$\frac{1}{3}\,\alpha = \frac{8}{3}\,\alpha - 1 \quad \text{or} \quad \alpha = \frac{3}{7} . \tag{3.6}$$

The first and second equation (2.1) now become

$$u_1^*\,\frac{\partial u_1^*}{\partial r^*} + w_1^*\,\frac{\partial u_1^*}{\partial z^*} = -\frac{\partial p_1^*}{\partial r^*} + \frac{\partial^2 u_1^*}{\partial z^{*2}} ,$$
$$u_1^*\,\frac{\partial v_1^*}{\partial r^*} + w_1^*\,\frac{\partial v_1^*}{\partial z^*} = \frac{\partial^2 v_1^*}{\partial z^{*2}} . \tag{3.7}$$

In the third equation (2.1) the terms $u(\partial \tilde{w}/\partial r)$, $\tilde{w}(\partial \tilde{w}/\partial \tilde{z})$ and $\partial^2 \tilde{w}/\partial \tilde{z}^2$ are all $O(\mathrm{Re}^{\alpha})$, while $\mathrm{Re}(\partial p/\partial \tilde{z}) = O(\mathrm{Re}^{2\alpha})$. Therefore, this last term must be zero, i.e.

$$\frac{\partial p_1^*}{\partial z^*} = 0 , \qquad p_1^* = p_1^*(r^*) . \tag{3.8}$$

The continuity equation is

$$\frac{\partial u_1^*}{\partial r^*} + \frac{\partial w_1^*}{\partial z^*} = 0\,. \tag{3.9}$$

4. The double deck

Summarizing, we have obtained the following results.

middle deck:

$$\begin{aligned}
\tilde{\psi}(r, \tilde{z}; \mathrm{Re}) &= \tfrac{1}{2}H(\tilde{z}) + \mathrm{Re}^{-1/7}E(r^*)H'(\tilde{z}) + \cdots,\\
u(r, \tilde{z}; \mathrm{Re}) &= \tfrac{1}{2}H'(\tilde{z}) + \mathrm{Re}^{-1/7}E(r^*)H''(\tilde{z}) + \cdots,\\
v(r, \tilde{z}; \mathrm{Re}) &= G(\tilde{z}) + 2\,\mathrm{Re}^{-1/7}E(r^*)G'(\tilde{z}) + \cdots,\\
\tilde{w}(r, \tilde{z}; \mathrm{Re}) &= -\mathrm{Re}^{2/7}E'(r^*)H'(\tilde{z}) + \cdots,\\
p(r, \tilde{z}; \mathrm{Re}) &= \mathrm{Re}^{-2/7}\left[-\frac{1}{2}\,E''(r^*)\int_{\tilde{z}}^{\infty}\{H'(\tilde{z})\}^2\,\mathrm{d}\tilde{z} + E_1(r^*)\right] + \cdots;
\end{aligned} \tag{4.1}$$

lower deck:

$$\begin{aligned}
\tilde{\psi}(r, \tilde{z}; \mathrm{Re}) &= \mathrm{Re}^{-2/7}\psi_1^*(r^*, z^*) + \cdots,\\
u(r, \tilde{z}; \mathrm{Re}) &= \mathrm{Re}^{-1/7}u_1^*(r^*, z^*) + \cdots,\\
v(r, \tilde{z}; \mathrm{Re}) &= 1 + \mathrm{Re}^{-1/7}v_1^*(r^*, z^*) + \cdots,\\
\tilde{w}(r, \tilde{z}; \mathrm{Re}) &= \mathrm{Re}^{1/7}w_1^*(r^*, z^*) + \cdots,\\
p(r, \tilde{z}; \mathrm{Re}) &= \mathrm{Re}^{-2/7}p_1^*(r^*) + \cdots.
\end{aligned} \tag{4.2}$$

The thickness of the middle deck is $O(\mathrm{Re}^{-1/2})$ and that of the lower deck is $O(\mathrm{Re}^{-9/14})$.

We still have to show that there is no upper deck. If it existed, there would be a pressure $O(\mathrm{Re}^{-2/7})$ in this potential region. But for $\tilde{z} \to \infty$ the functions $G(\tilde{z})$, $G'(\tilde{z})$, $H'(\tilde{z})$ and $H''(\tilde{z})$ all vanish, which means due to (4.1) that u and v both are $o(\mathrm{Re}^{-1/7})$. According to Bernoulli's law p then must be $o(\mathrm{Re}^{-2/7})$ and so there is no upper deck in the approximation we are investigating. This means

$$E_1(r^*) = 0\,. \tag{4.3}$$

For the pressure we then have

$$\begin{aligned}
&\textit{middle deck:}\quad p(r, \tilde{z}; \mathrm{Re}) = \mathrm{Re}^{-2/7}\left[-\frac{1}{2}\,E''(r^*)\int_{\tilde{z}}^{\infty}\{H'(\tilde{z})\}^2\,\mathrm{d}\tilde{z}\right] + \cdots,\\
&\textit{lower deck:}\quad p(r, \tilde{z}; \mathrm{Re}) = \mathrm{Re}^{-2/7}\left[-\frac{1}{2}\,E''(r^*)\int_{0}^{\infty}\{H'(\tilde{z})\}^2\,\mathrm{d}\tilde{z}\right] + \cdots.
\end{aligned} \tag{4.4}$$

Boundary conditions at $z^* \to \infty$ for the equations (3.7) are obtained from matching of the lower deck with the middle deck. For $\tilde{z} \to 0$ we obtain from (4.1)

$$u(r, \tilde{z}; \mathrm{Re}) = \{\tfrac{1}{2}\tilde{z} + \mathrm{Re}^{-1/7}E(r^*)\}H''(0) = \mathrm{Re}^{-1/7}\{\tfrac{1}{2}z^* + E(r^*)\}H''(0)\,,$$
$$v(r, \tilde{z}; \mathrm{Re}) = 1 + \{\tilde{z} + 2\,\mathrm{Re}^{-1/7}E(r^*)\}G'(0) = 1 + \mathrm{Re}^{-1/7}\{z^* + 2E(r^*)\}G'(0)\,.$$

Hence, for $z^* \to \infty$,

$$\begin{aligned} u_1^*(r^*, z^*) &\sim \{\tfrac{1}{2}z^* + E(r^*)\}H''(0)\,, \\ v_1^*(r^*, z^*) &\sim \{z^* + 2E(r^*)\}G'(0)\,. \end{aligned} \tag{4.5}$$

5. Renormalization of the equations of the lower deck

In order to get rid of some constants and to use the same notation as F.T. Smith [3], we introduce

$$\kappa = \tfrac{1}{2}H''(0)\,, \qquad \sigma = G'(0)\,, \qquad \gamma = \frac{1}{4}\int_0^\infty \{H'(\tilde{z})\}^2\,\mathrm{d}\tilde{z}\,, \tag{5.1}$$

$$\begin{aligned} &\psi_1^*(r^*, z^*) = \gamma^{2/7}\kappa^{-1/7}\Psi(X, Z)\,, \qquad u_1^*(r^*, z^*) = \gamma^{1/7}\kappa^{3/7}U(X, Z)\,, \\ &v_1^*(r^*, z^*) = \gamma^{1/7}\kappa^{-4/7}\sigma V(X, Z)\,, \qquad w_1^*(r^*, z^*) = \gamma^{-1/7}\kappa^{4/7}W(X, Z)\,, \\ &p_1^*(r^*) = \gamma^{2/7}\kappa^{6/7}P(X)\,, \qquad 2E(r^*) = \gamma^{1/7}\kappa^{-4/7}A(X)\,, \\ &r^* = \gamma^{3/7}\kappa^{-5/7}X\,, \qquad z^* = \gamma^{1/7}\kappa^{-4/7}Z\,. \end{aligned} \tag{5.2}$$

Equations (3.7), (3.9) and (4.4) then become

$$\begin{aligned} &U\frac{\partial U}{\partial X} + W\frac{\partial U}{\partial Z} = -\frac{\mathrm{d}P}{\mathrm{d}X} + \frac{\partial^2 U}{\partial Z^2}\,, \\ &U\frac{\partial V}{\partial X} + W\frac{\partial V}{\partial Z} = \frac{\partial^2 V}{\partial Z^2}\,, \end{aligned} \tag{5.3}$$

$$\frac{\partial U}{\partial X} + \frac{\partial W}{\partial Z} = 0\,, \qquad U = \frac{\partial \Psi}{\partial Z}\,, \qquad W = -\frac{\partial \Psi}{\partial X}\,, \qquad P = -\frac{\mathrm{d}^2 A}{\mathrm{d}X^2}\,, \tag{5.4}$$

while boundary conditions are

$$Z = 0\,, \qquad \Psi = 0 \begin{cases} \text{for } X < 0\,, & U = 0\,, V = 0\,, \\ \text{for } X > 0\,, & \dfrac{\partial U}{\partial Z} = 0\,, \dfrac{\partial V}{\partial Z} = 0\,. \end{cases} \tag{5.5}$$

From (4.5), (2.10) and the continuity equation in (5.3), we obtain

$$X \to -\infty: \quad U \sim Z\,, \qquad V \sim Z\,, \qquad W \to 0\,, \qquad P \to 0\,, \tag{5.6}$$

$$Z \to \infty: \quad U \sim Z + A(X)\,, \qquad V \sim Z + A(X)\,, \tag{5.7}$$

and, finally, using (4.4) and (2.9),

$$X \to \infty: \quad P \to 0\,. \tag{5.8}$$

6. Asymptotic behaviour of the solutions for $X \to -\infty$

In order to solve the system of equations (5.3), initial values for the solution at some large negative value of X must be available. It appears that only exponential decrease of the various functions U, W and P for $X \to -\infty$ satisfies (5.3) and its boundary conditions. Therefore, we put as in [2]

$$U = Z + f'(Z)\,\mathrm{e}^{\lambda X}, \qquad P = b\mathrm{e}^{\lambda X}\ (\lambda > 0), \quad \text{so} \quad W = -\lambda f(Z)\,\mathrm{e}^{\lambda X}. \tag{6.1}$$

Substituting these expressions in the first equation (5.3) and neglecting $\mathrm{e}^{2\lambda X}$, we obtain

$$f'''(Z) - \lambda Z f'(Z) + \lambda f(Z) = \lambda b$$

or, with $Z = \lambda^{-1/3}\bar{Z}$,

$$f'''(\bar{Z}) - \bar{Z} f'(\bar{Z}) + f(\bar{Z}) = b. \tag{6.2}$$

Differentiation of (6.2) yields

$$f^{(4)}(\bar{Z}) - \bar{Z} f''(\bar{Z}) = 0. \tag{6.3}$$

According to [6] the solutions $f''(\bar{Z})$ of (6.3) are the Airy functions $\mathrm{Ai}(\bar{Z})$ and $\mathrm{Bi}(\bar{Z})$. Since $\mathrm{Bi}(\bar{Z})$ increases exponentially with $\bar{Z} \to \infty$, we must have

$$f''(\bar{Z}) = c\,\mathrm{Ai}(\bar{Z}).$$

With the boundary conditions $f(0) = 0$ and $f'(0) = 0$, it follows from (6.2) that $f'''(0) = b$. Then

$$f''(\bar{Z}) = b\,\frac{\mathrm{Ai}(\bar{Z})}{\mathrm{Ai}'(0)}. \tag{6.4}$$

The integration of (6.2) has to be performed from a large value of $\bar{Z}$ to $\bar{Z} = 0$, since otherwise the function $\mathrm{Bi}(\bar{Z})$ creeps into the solution. Since, see [6], $\int_0^\infty \mathrm{Ai}(\bar{Z})\,\mathrm{d}\bar{Z} = \frac{1}{3}$, we have

$$f'(\bar{Z}) \to \frac{1}{3}\,\frac{b}{\mathrm{Ai}'(0)} \quad \text{for } \bar{Z} \to \infty. \tag{6.5}$$

For large values of $\bar{Z}$, f and f' are much larger than f'' and f'''. This leads to a loss of significant digits when integrating (6.2). Therefore we replace f by the function

$$g(\bar{Z}) = \frac{\mathrm{Ai}'(0)}{b}\,f(\bar{Z}) + c_1 + c_2\bar{Z} \tag{6.6}$$

and choose c_1 and c_2 in such a way that also g and g' vanish for $\bar{Z} \to \infty$. Using (6.5) it follows that $c_2 = -\frac{1}{3}$ and, by elimination of f from (6.2) and (6.6), we obtain $c_1 = -\mathrm{Ai}'(0)$ and

$$g'''(\bar{Z}) - \bar{Z} g'(\bar{Z}) + g(\bar{Z}) = 0. \tag{6.7}$$

To initialize the integration at some large value of $\bar{Z}$ we take, see [6],

$$g''(\bar{Z}) = \mathrm{Ai}(\bar{Z}) \sim \frac{e^{-\zeta}}{2\sqrt{\pi}\bar{Z}^{1/4}}\left(1 - \frac{15}{216}\zeta^{-1} + \cdots\right),$$

$$g'(\bar{Z}) \sim -\frac{e^{-\zeta}}{2\sqrt{\pi}\bar{Z}^{3/4}},$$

$$g(\bar{Z}) = \bar{Z}g'(\bar{Z}) - \mathrm{Ai}'(\bar{Z}) \sim \bar{Z}g'(\bar{Z}) + \frac{e^{-\zeta}\bar{Z}^{1/4}}{2\sqrt{\pi}}\left(1 + \frac{21}{216}\zeta^{-1} + \cdots\right).$$

where $\zeta = \frac{2}{3}\bar{Z}^{3/2}$.

$g'(\bar{Z})$, which is relatively the least accurate, is used as shooting parameter in order to make $f'(0) = 0$ ($g'(0) = -\frac{1}{3}$). It turned out that $g(0)$ and $g''(0)$ agreed in all 15 digits with the values for $-\mathrm{Ai}'(0)$ and $\mathrm{Ai}(0)$ available in [6]. As initial value for $\bar{Z}$ we took $\bar{Z} = \lambda^{1/3}Z_{max}$ with $Z_{max} = 12.5$.

We still have to determine the value of λ. From (5.4), (6.1) and (5.7) it follows that

$$X \to -\infty, \qquad A(X) \sim -\frac{b}{\lambda^2}e^{\lambda X} \quad \text{and} \quad f'(\infty) = -\frac{b}{\lambda^2}. \tag{6.8}$$

But from (6.5) we have

$$f'(\bar{Z}) \to \frac{1}{3}\lambda^{1/3}\frac{b}{\mathrm{Ai}'(0)} \quad \text{for } \bar{Z} \to \infty.$$

Combination of these results leads to

$$\lambda = \{-3\,\mathrm{Ai}'(0)\}^{3/7} = 0.897238191. \tag{6.9}$$

The remaining unknown parameter b has to be determined by a shooting procedure in order to satisfy the boundary condition (5.8).

For the determination of initial conditions for V at a large negative value of X, we proceed as follows. Put

$$V = Z + g_1(Z)e^{\lambda X}. \tag{6.10}$$

Together with (6.1) we substitute this in the second equation (5.3) and neglect $e^{2\lambda X}$. Then

$$g_1''(Z) - \lambda Z g_1(Z) = -\lambda f(Z)$$

or

$$g_1''(\bar{Z}) - \bar{Z}g_1(\bar{Z}) = -\lambda^{1/3}f(\bar{Z}). \tag{6.11}$$

Boundary conditions are

$$g_1(0) = 0 \quad \text{and, as follows from (5.7) and (6.8),} \quad g_1(\infty) = -\frac{b}{\lambda^2}. \tag{6.12}$$

The solution of (6.11) can be written as

$$g_1(\bar{Z}) = c\,\mathrm{Ai}(\bar{Z}) + bh(\bar{Z}) , \tag{6.13}$$

where the first term is the complementary function and the second term a particular integral.

Substitution of (6.13) into (6.11) leads to the differential equation for $h(\bar{Z})$,

$$h''(\bar{Z}) - \bar{Z}h(\bar{Z}) = -\frac{\lambda^{1/3}}{b} f(\bar{Z}) .$$

Elimination of $f(\bar{Z})$ with the aid of (6.6) gives

$$h''(\bar{Z}) - \bar{Z}h(\bar{Z}) = -\frac{\lambda^{1/3}}{\mathrm{Ai}'(0)} \left\{ g(\bar{Z}) + \mathrm{Ai}'(0) + \frac{1}{3}\bar{Z} \right\} . \tag{6.14}$$

This equation can be split into three equations. Using (6.9) we obtain

$$h_1''(\bar{Z}) - \bar{Z}h_1(\bar{Z}) = \frac{3}{\lambda^2} g(\bar{Z}) \quad \text{with solution} \quad h_1(\bar{Z}) = -\frac{3}{\lambda^2} g'(\bar{Z}) ,$$

$$h_2''(\bar{Z}) - \bar{Z}h_2(\bar{Z}) = \frac{1}{\lambda^2} \bar{Z} \quad \text{with solution} \quad h_2(\bar{Z}) = -\frac{1}{\lambda^2} ,$$

$$h_3''(\bar{Z}) - \bar{Z}h_3(\bar{Z}) = -\lambda^{1/3} \quad \text{with solution} \quad h_3(\bar{Z}) = \lambda^{1/3}\pi\,\mathrm{Gi}(\bar{Z}) ,$$

where $\mathrm{Gi}(\bar{Z})$ is the related Airy function satisfying the standard equation (10.4.55) of [6], viz.

$$w''(\bar{Z}) - \bar{Z}w(\bar{Z}) = -\pi^{-1} \tag{6.15}$$

with prescribed values for $w(0)$ and $w'(0)$. In order to evaluate $\mathrm{Gi}(\bar{Z})$, we have to integrate this equation from some large value of $\bar{Z}$, say $\bar{Z}_{\max} = 12.5\lambda^{1/3}$, to $\bar{Z} = 0$. Initial values $w(\bar{Z}_{\max})$ and $w'(\bar{Z}_{\max})$ can be obtained from the asymptotic series for $\mathrm{Gi}(\bar{Z})$ which is

$$\mathrm{Gi}(\bar{Z}) \sim \pi^{-1} \sum_{j=0}^{\infty} \frac{a_j}{\bar{Z}^{3j+1}} , \quad \bar{Z} \to \infty , \tag{6.16}$$

where $a_0 = 1$ and $a_{j+1} = (3j+1)(3j+2)a_j$ for $j \geq 0$.

The series (6.16) is semi-convergent and, hence, it produces only values of restricted accuracy. Therefore, the initial values taken for $w(\bar{Z}_{\max})$ and $w'(\bar{Z}_{\max})$ are contaminated by contributions of $\mathrm{Ai}(\bar{Z})$ and $\mathrm{Bi}(\bar{Z})$, that is,

$$w(\bar{Z}_{\max}) = \mathrm{Gi}(\bar{Z}_{\max}) + c_1\,\mathrm{Ai}(\bar{Z}_{\max}) + c_2\,\mathrm{Bi}(\bar{Z}_{\max}) ,$$

$$w'(\bar{Z}_{\max}) = \mathrm{Gi}'(\bar{Z}_{\max}) + c_1\,\mathrm{Ai}'(\bar{Z}_{\max}) + c_2\,\mathrm{Bi}'(\bar{Z}_{\max}) .$$

The coefficient c_2 can be calculated by integrating (6.15) to a value, say $2\bar{Z}_{\max}$, where the exponentially increasing term $c_2\,\mathrm{Bi}(\bar{Z})$ is extremely dominant since the other terms decrease. Dividing by the value of $\mathrm{Bi}(2\bar{Z}_{\max})$ we obtain c_2. We remove the terms with c_2 from the initial values for $w(\bar{Z}_{\max})$ and $w'(\bar{Z}_{\max})$. That these values still contain a term with $\mathrm{Ai}(\bar{Z})$ and $\mathrm{Ai}'(\bar{Z})$ is not at all harmful, since according to (6.13) a term with $\mathrm{Ai}(\bar{Z})$ should anyhow be added to $h(\bar{Z})$ in order to make $g_1(0) = 0$. This last condition allows us to calculate c in (6.13) and then to compute $g_1(\bar{Z})$.

7. Asymptotic behaviour of the solution for $X \to +\infty$

It has been remarked in Section 3 that for $X \to \infty$ the solution in the lower deck must match the inner Goldstein solution (3.3). Transformed to the coordinates introduced in Section 5 this solution is

$$U = \kappa^{-2/3} X^{1/3} f_0'(\eta_0)\,, \qquad V = \sigma^{-1}\kappa^{-2/3} X^{1/3} g_0(\eta_0)\,, \qquad \eta_0 = \frac{Z}{\kappa^{1/3} X^{1/3}}\,.$$

Then

$$\frac{\partial U}{\partial Z} = \kappa^{-1} f_0''(\eta_0)\,, \qquad \frac{\partial V}{\partial Z} = \sigma^{-1}\kappa^{-1} g_0'(\eta_0)\,.$$

The functions f_0 and g_0 satisfy the differential equations [1]

$$3f_0''' + 2f_0 f_0'' - f_0'^2 = 0\,, \tag{7.1}$$

$$3g_0'' + 2f_0 g_0' - f_0' g_0 = 0\,. \tag{7.2}$$

Since, according to (5.7), $\partial U/\partial Z$ and $\partial V/\partial Z$ are both equal to 1 for $Z \to \infty$, we have

$$\eta_0 \to \infty\,, \qquad f_0''(\eta_0) \to \kappa\,, \qquad g_0'(\eta_0) \to \sigma\kappa\,.$$

Boundary conditions at $\eta_0 = 0$ are

$$f_0(0) = 0\,, \qquad f_0''(0) = 0\,, \qquad g_0'(0) = 0\,.$$

Equation (7.1) possesses the similarity property, which means that if we replace $f_0(\eta_0)$ by $cf_1(\eta_1)$, with $\eta_1 = c\eta_0$, the equation remains invariant. Then

$$f_0'(\eta_0) \to c^2 f_1'(\eta_1)\,, \qquad f_0''(\eta_0) \to c^3 f_1''(\eta_1)\,, \qquad f_0'''(\eta_0) \to c^4 f_1'''(\eta_1)\,.$$

As boundary conditions for f_1 we take

$$f_1(0) = 0\,, \qquad f_1'(0) = 1\,, \qquad f_1''(0) = 0\,,$$

which implies that the boundary-value problem for f_0 is transformed into an initial-value problem for f_1. Numerical solution leads to $f_1''(\infty) = 0.489094382$. Since $f_0''(\infty) \to \kappa$, the value of c is determined by

$$c = \left\{\frac{\kappa}{f_1''(\infty)}\right\}^{1/3}.$$

Then $f_0'(0) = 1.610911012\kappa^{2/3}$ and $f_0'(\eta_0) \sim \kappa\eta_0 + 0.891998003\kappa^{2/3}$ for $\eta_0 \to \infty$. Hence, we have for $X \to \infty$

$$U(X, 0) \sim 1.610911012 X^{1/3} \quad \text{and} \quad U(X, Z) \sim Z + 0.891998003 X^{1/3} \quad \text{for } Z \to \infty\,.$$

This implies, using also (5.4),

$$A(X) \sim 0.891998003 X^{1/3}, \qquad P(X) \sim 0.198221778 X^{-5/3} \quad \text{for } X \to \infty. \tag{7.3}$$

It follows from (7.1) and (7.2) that $g_0 = c_1 f_0'$ is a solution of (7.2), where c_1 is an arbitrary constant. Hence, to $\eta_0 \to \infty$, $g_0'(\eta_0) \to \sigma\kappa$ corresponds $g_0(0) = 1.610911012\sigma\kappa^{2/3}$ and also

$$V(X, 0) \sim 1.610911012 X^{1/3} \quad \text{for } X \to \infty.$$

In order to investigate the next term in these asymptotic expansions we put

$$\Psi = \kappa^{-1/3} X^{2/3} f_0(\eta_0) + \kappa^{\alpha} X^{\beta} \varphi_0(\eta_0). \tag{7.4}$$

Substitution in the first equation (5.3) leads for φ_0 to the equation

$$\varphi_0''' + \tfrac{2}{3} f_0 \varphi_0'' - \beta f_0' \varphi_0' + \beta f_0'' \varphi_0 = 0, \tag{7.5}$$

provided the pressure term can be neglected. The boundary conditions are

$$\varphi_0(0) = 0, \qquad \varphi_0''(0) = 0 \quad \text{and} \quad \varphi_0''(\infty) = 0.$$

Hence, we have an eigenvalue problem. The largest eigenvalue β appears to be $\beta = -\frac{1}{3}$ and the pertaining eigensolution is

$$\varphi_0 = f_0 - \tfrac{1}{2}\eta_0 f_0'.$$

Since the terms in (5.3) which lead to (7.5) are $O(X^{\beta-1}) = O(X^{-4/3})$, it is permitted to neglect the pressure term since this is $O(X^{-8/3})$. The term with φ_0 in (7.4) implies that all asymptotic results for $X \to \infty$ have a relative error $O(X^{-1})$ corresponding to an origin shift.

8. Asymptotic behaviour of $V(X, Z)$ for $Z \to \infty$

The approximation

$$U \to Z + A(Z) \quad \text{for } Z \to \infty$$

contains an exponentially small error as can be seen from differentiation of (7.1) which gives

$$3 f_0^{(4)} + 2 f_0 f_0''' = 0.$$

However, $V \to Z + A(X)$ contains an error which is only algebraically small. Hence, we would make an important error, if we would use this approximation for a finite value of Z.

Before concentrating on the expansion of V, we have to derive the expansion for W. It follows from the continuity equation in (5.3) that this is

$$W \to -ZA'(X) + D(X), \quad Z \to \infty.$$

The function $D(X)$ can be determined by substitution in the first equation (5.3). This leads to

$$D(X) = -P' - AA' .$$

Thus $W \to -\{Z + A(X)\}A'(X) - P'(X)$. Putting $Z + A(X) = Y$, we have $U = Y$, $W = -YA'(X) - P'(X)$ and we take

$$V \sim Y + \frac{\alpha_1(X)}{Y} + \frac{\alpha_2(X)}{Y^2} + \cdots , \quad Y \to \infty \tag{8.1}$$

as asymptotic expansion for V.

Transforming the second equation (5.3) from the coordinates (X, Z) to (X, Y) and using the expressions for U and W, we obtain

$$Y \frac{\partial V}{\partial X} - \frac{\mathrm{d}P}{\mathrm{d}X} \frac{\partial V}{\partial Y} = \frac{\partial^2 V}{\partial Y^2} . \tag{8.2}$$

Substituting (8.1) into (8.2) and equating all powers of Y^{-1} to zero, the result after integration with respect to X, is

$$\alpha_1 = P , \qquad \alpha_2 = 0 , \qquad \alpha_3 = -\tfrac{1}{2}P^2 , \qquad \alpha_4 = -2A' , \qquad \alpha_5 = \tfrac{1}{2}P^3 . \tag{8.3}$$

The asymptotic expansion for V then becomes

$$V \sim Y + \frac{P}{Y} - \frac{P^2}{2Y^3} - \frac{2A'}{Y^4} + \frac{P^3}{2Y^5} + \cdots , \qquad Y = Z + A(X) \to \infty \tag{8.4}$$

For $X \to -\infty$ and neglecting terms of order $e^{2\lambda X}$, this result is in agreement with the result obtained in Section 6.

9. The numerical solution of the equations of the lower deck

After elimination of W, equations (5.3) and (5.4) are written as

$$\begin{aligned} &\frac{\partial \Psi}{\partial Z} = U , \qquad U \frac{\partial U}{\partial X} - \frac{\partial \Psi}{\partial X} \frac{\partial U}{\partial Z} = -\frac{\mathrm{d}P}{\mathrm{d}X} + \frac{\partial^2 U}{\partial Z^2} , \\ &B = \frac{\mathrm{d}A}{\mathrm{d}X} , \qquad P = -\frac{\mathrm{d}B}{\mathrm{d}X} . \end{aligned} \tag{9.1}$$

This is a set of equations with only first-order derivatives in X. Unknowns are $A(X)$, $B(X)$, $P(X)$, $U(X, Z)$ and $\Psi(X, Z)$. A fifth equation is given by the boundary condition (5.7),

$$Z \to \infty , \qquad U \to Z + A(X) .$$

In Z-direction we apply the transformation

$$Z = Z_{\max} \frac{\sinh \beta\mu}{\sinh \beta} \quad \text{with} \quad Z_{\max} = 12.5 \quad \text{and} \quad \beta = 5 . \tag{9.2}$$

In the finite difference method which is used to solve (9.1) we take $\mu_j = jh$, $h = 1/n$ and $j = 0, 1, \ldots, n$. This gives a greater density of points near $Z = 0$. In X-direction we use a transformation $X = f(\xi)$ with $f(\xi) = 10\xi$ for $X < 0$, $f(\xi) = \xi^3$ for $0 < X < 8$ and $f(\xi) = 12\xi - 16$ for $X > 8$. Moreover we replace the variable U by a variable T defined by

$$U = Z + T . \tag{9.3}$$

The equations (9.1) then are written as

$$\begin{aligned}
&\frac{\partial \Psi}{\partial Z} - Z - T = 0 , \\
&(Z + T) \frac{\partial T}{\partial \xi} - \frac{\partial \Psi}{\partial \xi} \left(1 + \frac{\partial T}{\partial \mu} \frac{d\mu}{dZ}\right) + \frac{dP}{d\xi} - f'(\xi)\left\{\frac{\partial^2 T}{\partial \mu^2} \left(\frac{d\mu}{dZ}\right)^2 + \frac{\partial T}{\partial \mu} \frac{d^2\mu}{dZ^2}\right\} = 0 , \\
&\frac{dA}{d\xi} - Bf'(\xi) = 0 , \\
&\frac{dB}{d\xi} + Pf'(\xi) = 0 , \\
&\mu_n = 1 , \qquad T(\xi, \mu_n) = A(\xi) , \\
&\mu_0 = 0 \begin{cases} \xi \leq 0 , & T(\xi, \mu_0) = 0 , \\ \xi > 0 , & 3T(\xi, \mu_0) - 4T(\xi, \mu_1) + T(\xi, \mu_2) - 4Z(\mu_1) + Z(\mu_2) = 0 . \end{cases}
\end{aligned} \tag{9.4}$$

The system (9.4) has been solved by the Crank–Nicolson method combined with an iterative Newton–Raphson procedure in order to cope with the non-linearity in the second equation. The first equation (9.4) has been discretized in the points $(\xi_i, (Z_{j-1} + Z_j)/2)$, $j = 1, 2, \ldots, n$, the second equation in the points $((\xi_{i-1} + \xi_i)/2, \mu_j)$, $j = 1, 2, \ldots, n$ and the third and fourth equations in the points $(\xi_{i-1} + \xi_i)/2$. Together with the fifth equation these are $2n + 3$ equations for $2n + 3$ unknowns at each value of ξ_i provided $\xi_i \leq 0$.

If $\xi_i > 0$, $T(\xi_i, \mu_0)$ is also unknown. Its value is obtained with the aid of the boundary condition $\partial U/\partial Z(X, 0) = 0$ which leads to the last equation of the system (9.4).

We now can perform the integration in X-direction, starting for some large negative value of X_0 with initial values derived in Section 6. The quantity b is unknown and has to be determined by a shooting procedure in such a way that $P \to 0$ for $X \to \infty$, equation (5.8).

After having obtained the correct value for b, we include the equation for V from (5.3). This is done as follows

$$V = Z + S ,$$

$$(Z + T)\frac{\partial S}{\partial \xi} - \frac{\partial \Psi}{\partial \xi}\left(1 + \frac{\partial S}{\partial \mu}\frac{d\mu}{dZ}\right) - f'(\xi)\left\{\frac{\partial^2 S}{\partial \mu^2}\left(\frac{d\mu}{dZ}\right)^2 + \frac{\partial S}{\partial \mu}\frac{d^2\mu}{dZ^2}\right\} = 0 ,$$

$$\mu_n = 1 , \qquad S(\xi, \mu_n) = A + \frac{P}{Y} - \frac{P^2}{2Y^3} - \frac{2A'}{Y^4} + \frac{P^3}{2Y^5} , \qquad Y = Z_{\max} + A(X) , \tag{9.5}$$

$$\mu_0 = 0 \begin{cases} \xi \leq 0 , & S(\xi, \mu_0) = 0 , \\ \xi > 0 , & 3S(\xi, \mu_0) - 4S(\xi, \mu_1) + S(\xi, \mu_2) - 4Z(\mu_1) + Z(\mu_2) = 0 . \end{cases}$$

Initial values for $S(X_0)$ are again obtained from Section 6.

10. The torque

The torque which has to be exerted on the disk in order to maintain the constant angular velocity Ω is

$$M = -2\pi\rho a^5 \Omega^2 \operatorname{Re}^{-1/2} \int_0^1 r^2 \left.\frac{\partial v}{\partial \tilde{z}}\right|_0 dr .$$

This torque can be split into a classical part,due to the term $v = rG(\tilde{z})$, and a part due to the modification of the double deck. The classical part is equal to

$$M_d = -\tfrac{1}{2}\sigma\pi\rho a^5 \Omega^2 \operatorname{Re}^{-1/2} , \tag{10.1}$$

where σ is given by (5.1). With the aid of (3.1) and (5.2) we obtain for the contribution due to the double deck

$$M = -2\pi\rho a^5 \Omega^2 \operatorname{Re}^{-13/14} \sigma\gamma^{3/7}\kappa^{-5/7} \int_{-\infty}^{0} \left.\frac{\partial V}{\partial Z}\right|_0 dX .$$

Substituting $\partial V/\partial Z = 1 + \partial S/\partial Z$, the term 1 belongs to the classical part of the torque which, of course, is also present in the double-deck region. Hence, the additional torque is

$$M_{add} = -2\pi\rho a^5 \Omega^2 \operatorname{Re}^{-13/14} \sigma\gamma^{3/7}\kappa^{-5/7} \int_{-\infty}^{0} \left.\frac{\partial S}{\partial Z}\right|_0 dX . \tag{10.2}$$

The integral will be divided as

$$\int_{-\infty}^{0} = \int_{-\infty}^{X_0} + \int_{X_0}^{0} ,$$

where X_0 is again the negative value of X where the numerical integration of the set (9.4) starts. For $X < X_0$ we use (6.10). Then

$$\int_{-\infty}^{X_0} \left.\frac{\partial S}{\partial Z}\right|_0 dX = g_1'(0) \int_{-\infty}^{X_0} e^{\lambda X} dX = \frac{1}{\lambda} g_1'(0) e^{\lambda X_0} ,$$

where $g_1'(0)$ is known from Section 6, and

$$\int_{X_0}^{0} \frac{\partial S}{\partial Z}\bigg|_0 \mathrm{d}X = \frac{1}{2h} \frac{\mathrm{d}\mu}{\mathrm{d}Z}\bigg|_0 \int_{X_0}^{0} \{4S(X, \mu_1) - S(X, \mu_2)\}\, \mathrm{d}X ,$$

where the last integral is calculated by the trapezoidal rule.

11. Numerical results

The values of the constants $\kappa = \frac{1}{2}H''(0)$ and $\sigma = G'(0)$ have already been obtained in [1] by integration of the system of differential equations for G and H. In order to calculate γ, we put

$$\gamma = \gamma_0(\infty) \quad \text{with} \quad \gamma_0(\tilde{z}) = \frac{1}{4} \int_0^{\tilde{z}} \{H'(\tilde{z})\}^2\, \mathrm{d}\tilde{z} ,$$

and we add the differential equation

$$\frac{\mathrm{d}\gamma_0}{\mathrm{d}\tilde{z}} = \frac{1}{4} \{H'(\tilde{z})\}^2$$

to the equations for G and H. The results are

$$\kappa = 0.510232619 , \qquad \sigma = -0.615922014 , \qquad \gamma = 0.054098089 .$$

After having solved (6.7) for $g(\bar{Z})$ and then (6.6) for $f(\bar{Z})$, we could take initial values for U, W and P at $X_0 = -14$ with an estimated value of b. The value $X_0 = -14$ is sufficiently large negative, since the neglected term $e^{2\lambda X_0}$ is $O(10^{-11})$ there. The discretized values of $\bar{Z}$ are determined by (9.2) together with $\bar{Z} = \lambda^{1/3} Z$.

The integration of the system (9.4) has been performed with different steps in μ and ξ. The step in μ is determined by the number n and the step in ξ is taken equal to k. The value of b is determined in such a way that the deviation of P from its asymptotic behaviour $P \to 0$ for $X \to \infty$ is delayed as long as possible.

By determining b in 15 significant digits, the deviation became only perceptible for $X > 13$. The value of b strongly depends upon n, k and the precise form of the discretisation scheme. In 6 digits some of the results are

$$n = 40 , \quad k = 0.01: \quad b = -0.264212 ,$$

$$n = 80 , \quad k = 0.005: \quad b = -0.267146 ,$$

$$n = 160 , \quad k = 0.0025: \quad b = -0.267878 ,$$

which shows very well an error in b of $O(k^2)$.

Table 1 shows some values of A and P for larger values of X. The table is based upon the calculations with the finest mesh. The values in the columns A^+ and P^+ are obtained with a value of b for which $P \to +\infty$, while in the case of A^- and P^- the value of b is such that $P \to -\infty$ for $X \to \infty$.

Table 1

X	A^+	A^-	$X^{-1/3}A^+$	P^+	P^-
8.0	1.750394167	1.750394168	0.875197	0.007101	0.007101
9.2	1.839012481	1.839012487	0.877651	0.005466	0.005466
10.4	1.919695900	1.919695943	0.879471	0.004375	0.004375
11.6	1.994042144	1.994042433	0.880876	0.003601	0.003600
12.8	2.063179267	2.063181289	0.881997	0.003029	0.003024
14.0	2.127937037	2.127951677	0.882910	0.002605	0.002565

According to Section 7 the asymptotic behaviour of A and P is given by

$$A \sim c_1X^{1/3} + c_2X^{-2/3} + c_3X^{-5/3} + \cdots , \quad \text{with} \quad c_1 = 0.891998003 ,$$

$$P \sim \tfrac{2}{9}c_1X^{-5/3} - \tfrac{10}{9}c_2X^{-8/3} - \tfrac{40}{9}c_3X^{-11/3} + \cdots .$$

In Table 1 we have also given the value of $X^{-1/3}A^+$. The difference with the limit value c_1 is mainly due to the next term in the expansion. In order to find better numerical approximations for the values of c_1, c_2 and c_3, we have interpolated for different meshes the values of A^+ in three points by the first three terms of the asymptotic expansion. The resulting approximations for c_1, c_2 and c_3 are given in a few digits in Table 2, and we observe that the numerical value for c_1 comes close to its theoretical value. The values for c_2 and c_3 are of course less accurate. We further remark that the interpolation errors appear to be extremely small, namely less than 10^{-6} in A. With the same values for c_1, c_2 and c_3 in the expansion for P we get errors less than 10^{-5}.

There is a difference between the present numerical values and those of F.T. Smith [3]. With initial values of $P = -106.10^{-7}$ and $P = -108.10^{-7}$, which Smith takes at $X_0 = -12.2$, he finds for X large, the divergences $P \to +\infty$ and $P \to -\infty$, respectively. Our value of P at $X = -12.2$ is -47.10^7. The corresponding value of b would be -0.6 in Smith's calculations. A further difference is that Smith accepts as result beyond $X = 3$ the first term of the asymptotic series for P within 'graphical' accuracy. However, since the second term in the asymptotic series is only a factor X^{-1} smaller than the first one, this seems not to be justified.

Smith's calculations were performed with a two-zone scheme in the region $0 < X < 1$ in order to take into account the singularity in $\partial U/\partial X$ at $X = 0$, $Z = 0$. However, we also performed calculations with a two-zone scheme. These only modified the results for extremely small values of X, but only very slightly at large values of X. This is in agreement with results obtained in [1]. The use of a two-zone scheme cannot explain the differences with [3].

Using the numerical values for κ, σ and γ, the torque is obtained from (10.1) and (10.2) as

$$M \sim \rho a^5\Omega^2(1.2900\,\mathrm{Re}^{-1/2} + 0.2236\,\mathrm{Re}^{-13/14}) .$$

Table 2

		Interpolation in (9.2), (10.4), (11.6)			Interpolation in (10.4), (11.6), (12.8)		
n	k	c_1	c_2	c_3	c_1	c_2	c_3
40	0.01	0.8927	−0.1133	−0.1598	0.8928	−0.1157	−0.1466
80	0.005	0.8919	−0.1116	−0.1639	0.8920	−0.1146	−0.1480
160	0.0025	0.8917	−0.1112	−0.1651	0.8918	−0.1138	−0.1509

12. Behaviour near $X = 0$

From the one-zone scheme used, the behaviour of functions for $X \downarrow 0$ cannot be obtained with sufficient accuracy. Therefore, we investigate this behaviour semi-analytically, using results for $X \uparrow 0$. The investigation of $U(X, 0)$ and $P(X)$ for $X \downarrow 0$ is identical to that of Veldman [7], except that a different value for the shear stress at $X \uparrow 0$ must be employed. The investigation of $V(X, 0)$ is new.

For $X \uparrow 0$ we have

$$\Psi(0, Z) = R_1(Z)\,, \qquad V(0, Z) = R_2(Z)\,,$$

where $R_1(Z)$ and $R_2(Z)$ have been calculated according to Section 9,

$$Z \to \infty \begin{cases} R_1(Z) = \tfrac{1}{2}Z^2 + ZA(0) + A_1(0) + \text{exp. small}\,, \\ R_2(Z) = Z + A(0) + \text{algebraic decrease}\,, \end{cases}$$

$$Z \to 0 \begin{cases} U(0, Z) = \dfrac{\mathrm{d}R_1}{\mathrm{d}Z} = a_2 Z + \tfrac{1}{2}a_3 Z^2 + \cdots\,, \\ V(0, Z) = R_2 = b_2 Z + \tfrac{1}{2}b_3 Z^2 + \cdots\,, \end{cases}$$

with $a_2 = 1.3993$, $a_3 = -0.3216$, $b_2 = 1.1160$, $b_3 = 0$.

For $X > 0$ there exists in the lower deck again an inner and an outer Goldstein solution. The expansion of the inner solution for $X \downarrow 0$ begins as

$$\Psi(X, \eta) \sim X^{2/3} F_0(\eta) + \cdots\,, \qquad \eta = Z/X^{1/3}\,.$$

Substituting in the first equation (5.3), we obtain

$$\tfrac{1}{3}X^{-1/3}F_0'^2 - \tfrac{2}{3}X^{-1/3}F_0F_0'' = -\frac{\mathrm{d}P}{\mathrm{d}X} + X^{-1/3}F_0'''\,,$$

which means that $\mathrm{d}P/\mathrm{d}X$ may also be of order $X^{-1/3}$. Then

$$P(X) = P(0) + p_{2/3}X^{2/3}\,.$$

The boundary-value problem for the inner solution is

$$3F_0''' + 2F_0F_0'' - F_0'^2 = 2p_{2/3}\,, \qquad F_0(0) = 0\,, \qquad F_0''(0) = 0\,, \qquad F_0''(\infty) = a_2\,. \tag{12.1}$$

Substitution of $V(X, \eta) = X^{1/3}G_0(\eta)$ into the second equation (5.3) yields

$$3G_0'' + 2F_0G_0' - F_0'G_0 = 0\,, \qquad G_0'(0) = 0\,, \qquad G_0'(\infty) = b_2\,. \tag{12.2}$$

The expansions of the outer solutions are

$$X \downarrow 0 \begin{cases} \Psi(X, Z) \sim R_1(Z) + X^{1/3}F_{1/3}(Z) + X^{2/3}F_{2/3}(Z) + \cdots\,, \\ V(X, Z) \sim R_2(Z) + X^{1/3}G_{1/3}(Z) + X^{2/3}G_{2/3}(Z) + \cdots\,. \end{cases} \tag{12.3}$$

Substitution in the first and second equations (5.3) leads to terms $O(X^{-2/3})$:

$$R_1'(Z)F_{1/3}'(Z) - R_1''(Z)F_{1/3}(Z) = 0\,,$$
$$R_1'(Z)G_{1/3}(Z) - R_2'(Z)F_{1/3}(Z) = 0\,. \qquad (12.4)$$

The first of these equations has the solution

$$F_{1/3}(Z) = CR_1'(Z)\,,$$

which gives

$$U(X, Z) = R_1'(Z) + CX^{1/3}R_1''(Z) + O(X^{2/3})\,.$$

For $Z \to \infty$, $R_1'(Z) \to Z + A(0)$ and $R_1''(Z) \to 1$ and thus

$$U(X, Z) \to Z + A(0) + CX^{1/3}\,.$$

On the other hand, according to (5.7)

$$U(X, Z) \to Z + A(X)\,.$$

However, $A(X)$ cannot contain a term $O(X^{1/3})$ in its expansion for small X, since this would lead with (5.4) to infinite pressure. Thus $C = 0$ and $F_{1/3}(Z) = 0$. It follows from the second equation (12.4) that then also $G_{1/3}(Z) = 0$.

We now consider terms $O(X^{-1/3})$ in the result obtained after substitution of (12.3) into the first equation (5.3). These are

$$R_1'(Z)F_{2/3}'(Z) - R_1''(Z)F_{2/3}(Z) = -p_{2/3}$$

leading to

$$F_{2/3}(Z) = C_1R_1'(Z) + \text{particular integral}\,.$$

Since $A(X)$ neither contains a term $O(X^{2/3})$, also C_1 must be equal to 0. For $Z \to \infty$ the particular integral in $F_{2/3}(Z)$ approaches the value $p_{2/3}$. Hence the function $F_{2/3}(Z)$ does not vanish, but it will not give a contribution to $U(X, Z)$ for $Z \to \infty$.

Since the outer expansion of $\Psi(X, Z)$ does not contain a term $O(X^{1/3})$, it follows from matching that the expansion of the inner solution $F_0(\eta)$ for $\eta \to \infty$ cannot contain a term linear in η (see also Daniels [8]). Hence

$$\eta \to \infty\,, \qquad F_0(\eta) = \tfrac{1}{2}a_2(\eta^2 + A_2) + \text{exp. small}\,. \qquad (12.5)$$

The term with A_2 exists because $F_{2/3}(Z) \neq 0$. By substitution of (12.5) into the equation (12.1) for $F_0(\eta)$ it follows that

$$p_{2/3} = \tfrac{1}{2}a_2^2A_2\,. \qquad (12.6)$$

By differentiation of (12.1) to η, we get rid of $p_{2/3}$ and obtain the following problem for $F_0(\eta_0)$:

$$3F_0^{(4)} + 2F_0F_0''' = 0\,, \qquad F_0(0) = 0\,, \qquad F_0''(0) = 0\,, \qquad F_0''(\infty) = a_2\,,$$

$$\eta_0 \to \infty\,, \qquad F_0(\eta_0) = \tfrac{1}{2}a_2(\eta_0^2 + A_2) + \text{exp. small}\,, \qquad \eta_0 = Z/X^{1/3}\,.$$

We apply the same similarity transformation as in Section 7 for $f_0(\eta_0)$, viz.

$$\eta_1 = c\eta_0\,, \qquad F_0(\eta_0) = cF_1(\eta_1)\,, \qquad F_0'(\eta_0) = c^2F_1'(\eta_1)\,, \quad \text{etc.} \tag{12.7}$$

We solve the equation $3F_1^{(4)} + 2F_1F_1''' = 0$ with boundary conditions $F_1(0) = 0$, $F_1'(0) = 1$, $F_1''(0) = 0$ and determine $F_1'''(0)$ in such a way that $F_1(\eta_1)$ does not contain a term linear in η_1 for $\eta_1 \to \infty$, that is,

$$\eta_1 \to \infty\,, \qquad F_1'(\eta_1) = \eta_1 F_1''(\eta_1)\,.$$

It appears that

$$F_1'''(0) = 0.839099815\,, \qquad F_1''(\infty) = 1.172886565\,.$$

Then

$$c = \left\{\frac{F_0''(\infty)}{F_1''(\infty)}\right\}^{1/3} = 0.948232121a_2^{1/3}\,,$$

$$F_0'(0) = c^2F_1'(0) = 0.899144155a_2^{2/3}\,,$$

$$\eta_1 \to \infty\,, \qquad F_1(\eta_1) \sim \tfrac{1}{2}\eta_1^2F_1''(\infty) + 0.646822758\,.$$

Substituting this in the relation

$$cF_1(\eta_1) \sim \frac{1}{2}\,a_2\left(\frac{\eta_1^2}{c^2} + A_2\right)$$

we obtain $A_2 = 1.226676231a_2^{-2/3} = 0.981$ and $p_{2/3} = 0.613338116a_2^{4/3} = 0.960$. Inserting also the value of a_2, we find

$$\begin{aligned} X \downarrow 0, \quad & U(X,0) \sim X^{1/3}F_0'(0) = 1.1249X^{1/3}\,, \\ & P(X) \sim -0.3109 + 0.960X^{2/3}\,, \\ X \uparrow 0, \quad & P(X) \sim P(0) + a_3X = -0.3109 - 0.3216X\,. \end{aligned}$$

After having obtained the function $F_0(\eta)$ and its derivatives, we consider equation (12.2), which is linear in $G_0(\eta)$. We solve this equation with boundary conditions

$$G_0(0) = 1\,, \qquad G_0'(0) = 0\,. \tag{12.8}$$

Since $G_{1/3}(Z)=0$, the expansion of $G_0(\eta)$ for large η does not contain a constant. It follows from (12.2) that $G_0(\eta)$ decreases algebraically for $\eta \to \infty$. By substitution in (12.2) the expansion of $G_0(\eta)$ appears to be

$$G_0(\eta) \sim b_2\left(\eta + \frac{b_3}{\eta} + \frac{b_4}{\eta^3} + \frac{b_5}{\eta^4} + \frac{b_6}{\eta^5} + \frac{b_7}{\eta^6} + \cdots\right), \tag{12.9}$$

where the constant term and the term with η^{-2} are absent, while

$$b_3 = \frac{1}{2}A_2\,,\qquad b_4 = -\frac{1}{8}A_2^2\,,\qquad b_5 = \frac{3A_2}{5a_2}\,,\qquad b_6 = \frac{1}{16}A_2^3\,,\qquad b_7 = -\frac{69A_2^2}{70a_2}\,.$$

By integrating (12.2) with boundary conditions (12.8) until $\eta = 10$, we find from (12.9) a value $b_2 = 0.7430$. Since in reality $b_2 = 1.1160$, the correct boundary conditions are

$$G_0(0) = 1.502\,,\qquad G_0'(0) = 0$$

and the expansion for V is

$$X \downarrow 0\,,\quad V(X,0) \sim 1.502X^{1/3}\,.$$

For the function $A(X)$ we have the expansions

$$X \uparrow 0\,,\quad A(X) \sim 0.3456 + 0.3218X + 0.1555X^2 + 0.0536X^3\,,$$

$$X \downarrow 0\,,\quad A(X) \sim 0.3456 + 0.3218X + 0.1555X^2 - 0.2160X^{8/3}\,.$$

Finally, it can be remarked that, for $X \downarrow 0$, $W(X)$ still is singular, although the singularity is like $X^{-1/3}$, which is weaker than the singularity $X^{-2/3}$ following from boundary-layer theory [1]. In order to remove also the singularity $O(X^{-1/3})$ we have to consider regions for which $r - 1 = O(\mathrm{Re}^{-\alpha})$ is smaller than $O(\mathrm{Re}^{-3/7})$. It follows from Section 2 that a further specific value for α is $\alpha = \frac{1}{2}$. It may be that there are still more specific values of α before attaining $\alpha = \frac{3}{4}$, where the complete Navier–Stokes equations have to be taken into account. This would be analogous to the situation at the trailing edge of a flat plate as described in [7].

13. Results in graphical form

Figure 1 shows P as a function of X, Fig. 2 shows A as function of X, Fig. 3 shows V for $Z = 0$ as function of X, and Fig. 4 shows V as function of Z for some values of X. Diagrams showing U for $Z = 0$ as function of X and U as function of Z for X-values are almost identical to the corresponding diagrams for V, which is the reason why they are omitted. Figure 5 shows $\int_{\tilde{z}}^{\infty} \{H'(\zeta)\}^2\, d\zeta$ as function of $\tilde{z}$, which determines the decrease of pressure in $\tilde{z}$-direction in the middle deck.

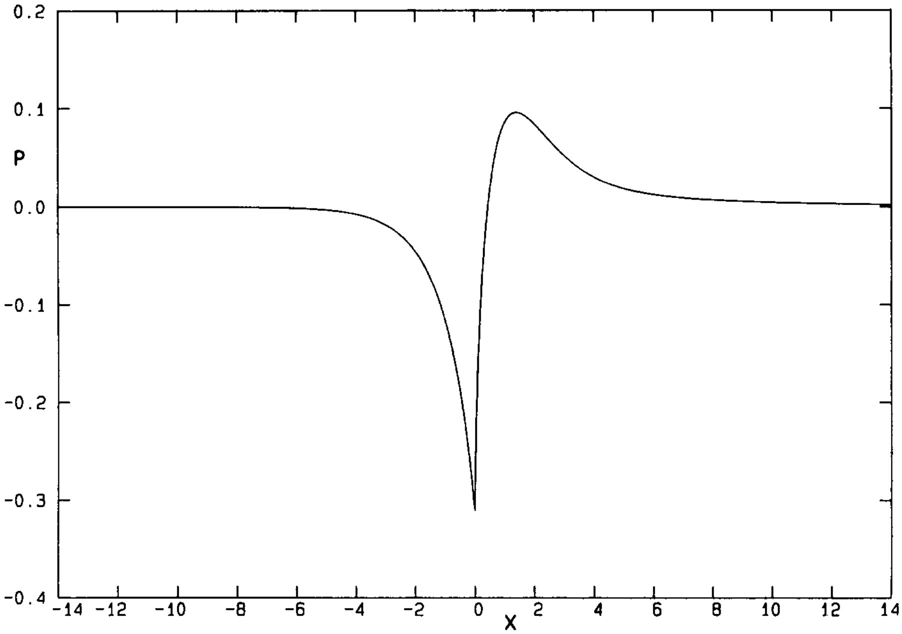

Fig. 1. The pressure in the lower deck as function of the radial coordinate.

$p = \mathrm{Re}^{-2/7}\gamma^{2/7}\kappa^{6/7}P(X), \quad r = 1 + \mathrm{Re}^{-3/7}\gamma^{3/7}\kappa^{-5/7}X,$
$X \to -\infty, \quad P(X) \sim -0.268\exp(-0.8972X),$
$X \uparrow 0, \quad P(X) \sim -0.3109 - 0.3216X,$
$X \downarrow 0, \quad P(X) \sim -0.3109 + 0.960X^{2/3},$
$X \to \infty, \quad P(X) \sim -0.1982X^{-5/3}.$

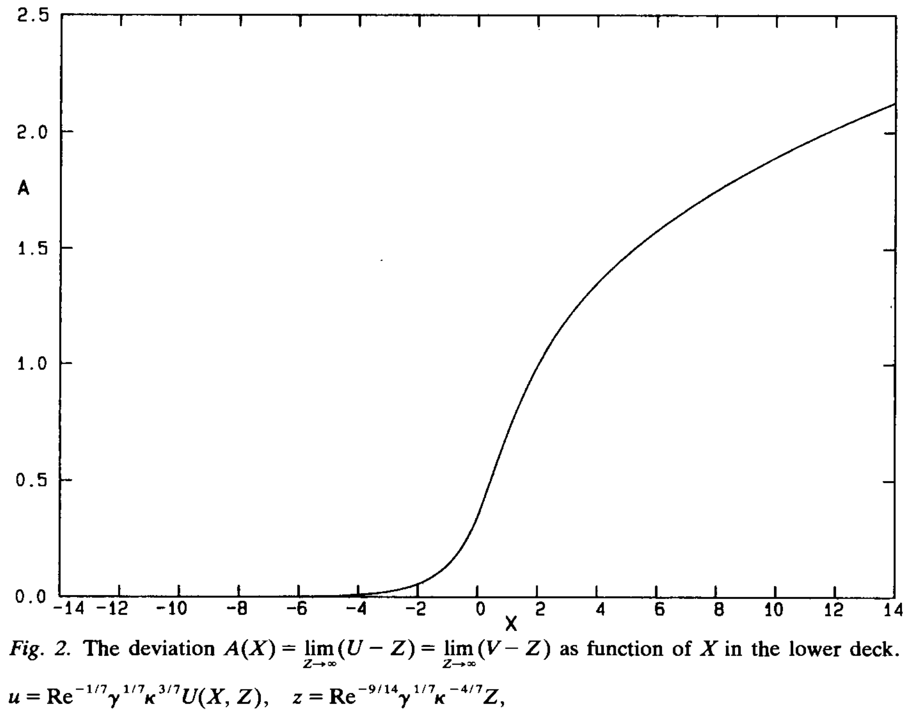

Fig. 2. The deviation $A(X) = \lim_{Z\to\infty}(U - Z) = \lim_{Z\to\infty}(V - Z)$ as function of X in the lower deck.

$u = \mathrm{Re}^{-1/7}\gamma^{1/7}\kappa^{3/7}U(X, Z), \quad z = \mathrm{Re}^{-9/14}\gamma^{1/7}\kappa^{-4/7}Z,$
$X \to -\infty, \quad A(X) \sim 0.333\exp(-0.8972X),$
$X \uparrow 0, \quad A(X) \sim 0.3456 + 0.3218X + 0.1555X^2 + 0.0536X^3,$
$X \downarrow 0, \quad A(X) \sim 0.3456 + 0.3218X + 0.1555X^2 - 0.2160X^{8/3},$
$X \to \infty, \quad A(X) \sim 0.8920X^{1/3}.$

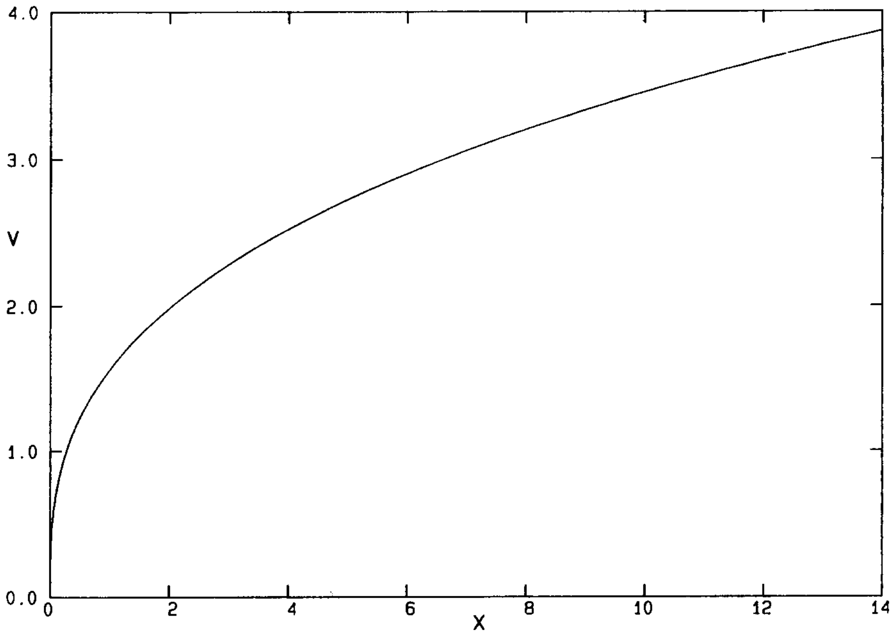

Fig. 3. The tangential velocity for $Z = 0$ as function of the radial coordinate.
$v = 1 + \mathrm{Re}^{-1/7}\gamma^{1/7}\kappa^{-4/7}\sigma V(X, Z)$,
$X \leq 0, \quad V = 0,$
$X \downarrow \hat{0}, \quad V(X, 0) \sim 1.502X^{1/3}, \quad U(X, 0) \sim 1.125X^{1/3},$
$X \to \infty, \quad V(X, 0) \sim 1.611X^{1/3}, \quad U(X, 0) \sim 1.611X^{1/3}.$

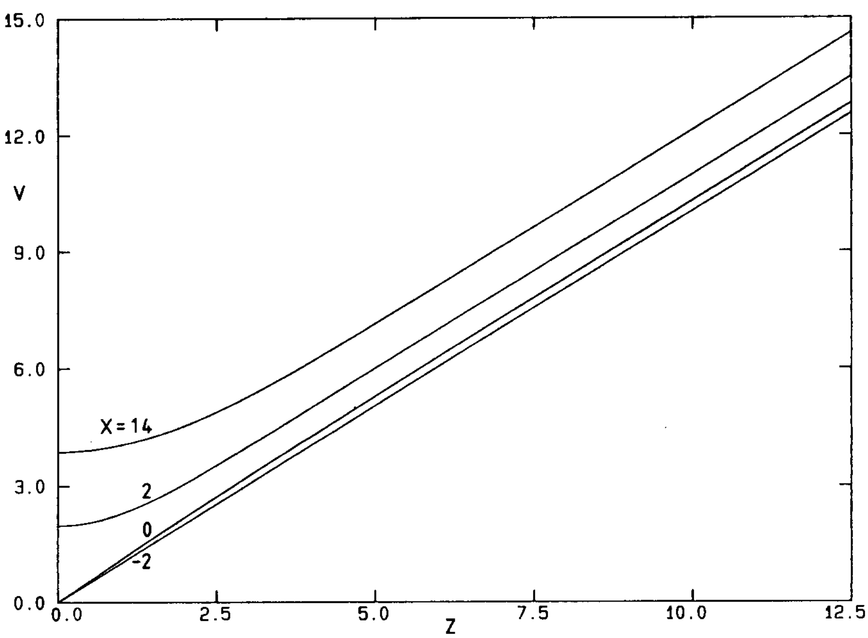

Fig. 4. The tangential velocity for some values of X as function of Z in the lower deck.
$v = 1 + \mathrm{Re}^{-1/7}\gamma^{1/7}\kappa^{-4/7}\sigma V(X, Z)$

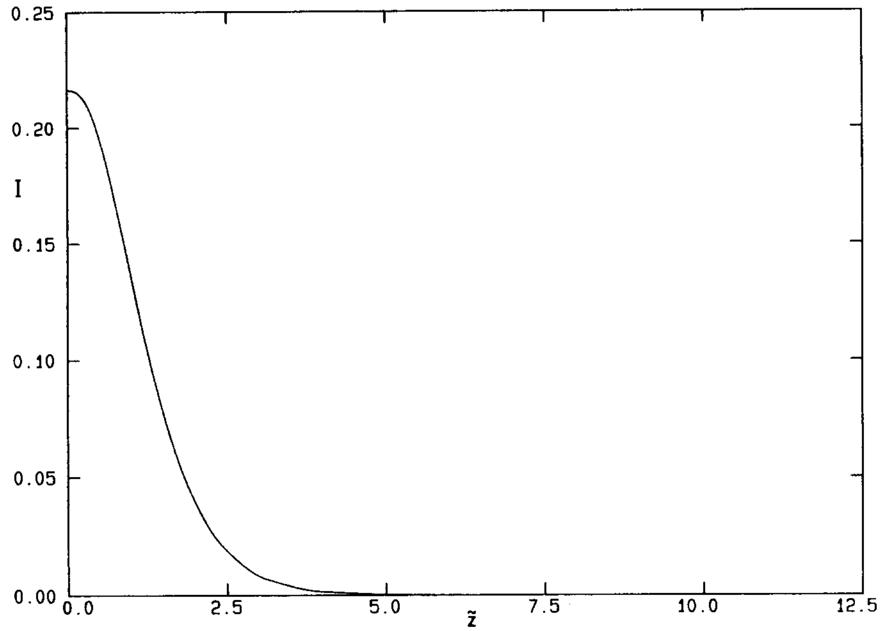

Fig. 5. The pressure decrease as function of the axial coordinate in the middle deck.

$z = \mathrm{Re}^{-1/2}\tilde{z}, \qquad p = \frac{1}{4}\mathrm{Re}^{-2/7}\gamma^{-5/7}\kappa^{6/7}P(X)I, \quad \text{where} \quad I = \int_{\tilde{z}}^{\infty} \{H'(\zeta)\}^2\,\mathrm{d}\zeta.$

14. Conclusions

The double deck extending over a length of $O(\mathrm{Re}^{-3/7})$ at both sides of the edge of a rotating disk has been investigated. In the middle deck of thickness $O(\mathrm{Re}^{-1/2})$ additional radial and tangential velocity components arise of $O(\mathrm{Re}^{-1/7})$, while the axial component is $O(\mathrm{Re}^{-3/14})$. In the usual boundary layer the orders of magnitude are Re^0, Re^0 and $\mathrm{Re}^{-1/2}$, respectively. The singularity $O(r-1)^{-2/3}$ of the classical axial velocity is removed in the middle deck of length $O(\mathrm{Re}^{-3/7})$ and replaced by a factor $O(\mathrm{Re}^{2/7})$, which increases the axial velocity to the level $O(\mathrm{Re}^{-3/14})$.

In the middle deck all velocity components and also the pressure have in axial direction a distribution which is independent of the radial coordinate r^*. Only the magnitude depends on r^*. The pressure is $O(\mathrm{Re}^{-2/7})$. This pressure is due to the change in radial velocity of $O(\mathrm{Re}^{-1/7})$ over the small distance of $O(\mathrm{Re}^{-3/7})$.

In the lower deck of thickness $O(\mathrm{Re}^{-9/14})$ the radial velocity is $O(\mathrm{Re}^{-1/7})$ and the tangential velocity is equal to that of the disk $O(\mathrm{Re}^0)$ with a correction $O(\mathrm{Re}^{-1/7})$. The axial velocity is $O(\mathrm{Re}^{-5/14})$ but it still contains a singularity $r^{*\,-1/3}$. In order to remove this singularity, regions smaller than $O(\mathrm{Re}^{-3/7})$ will have to be considered, in analogy to the situation at the trailing edge of a flat plate as described by Veldman [7].

Finally, the main purpose of the investigation was to calculate a further term in the expansion for $\mathrm{Re} \to \infty$ of the torque required for a steady rotation of the disk, and this leads to the result

$$M \sim \rho a^5 \Omega^2 (0.9675\mathrm{Re}^{-1/2} + 0.2236\,\mathrm{Re}^{-13/14})\,.$$

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